

# Density not realizable as the Jacobian determinant of a bilipschitz map\*

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## Abstract

Are every two separated nets in the plane bilipschitz equivalent? In the late 1990s, Burago and Kleiner and, independently, McMullen resolved this beautiful question negatively. Both solutions are based on a construction of a density function that is not realizable as the Jacobian determinant of a bilipschitz map. McMullen's construction is simpler than the Burago–Kleiner one, and we provide a full proof of its nonrealizability, which has not been available in the literature.

## 1 Introduction

**Non-equivalent separated nets and nonrealizable density.** We recall that a *separated net* in the plane is a set  $P \subset \mathbb{R}^2$  in which every two points have distance bounded below by some  $r > 0$  and the distance between any point in  $\mathbb{R}^2$  and the set  $P$  is bounded above by another constant  $R > 0$ . A simple example of a 1-separated 1-net is the integer lattice  $\mathbb{Z}^2$ .

The following fascinating question was first mentioned by Furstenberg in the 1960s and it appears in Gromov's book [Gro93]: *Are every two separated nets in the plane bilipschitz equivalent?*<sup>1</sup>

It was resolved negatively in the late 1990s by Burago and Kleiner [BK98] and independently by McMullen [McM98]. Both of the counterexamples are

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\*A preliminary version appeared in Czech as a part of the author's bachelor thesis [Kal12] at the Charles University in Prague in 2012.

<sup>1</sup>Let  $A, B \subseteq \mathbb{R}^2$ . We recall that a map  $\varphi: A \rightarrow B$  is *L-Lipschitz*, for a real number  $L > 0$ , if  $\|\varphi(\mathbf{a}) - \varphi(\mathbf{b})\| \leq L\|\mathbf{a} - \mathbf{b}\|$  for every  $\mathbf{a}, \mathbf{b} \in A$ . We say that  $\varphi$  is *L-bilipschitz* if both  $\varphi$  and  $\varphi^{-1}$  are *L-Lipschitz*, and we call  $\varphi$  *Lipschitz* or *bilipschitz* if it is *L-Lipschitz* or *L-bilipschitz*, respectively, for some  $L > 0$ . Two separated nets  $P$  and  $Q$  are bilipschitz equivalent if there is a bilipschitz bijection  $\varphi: P \rightarrow Q$ .

based on constructing a bounded density function  $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $\inf \rho > 0$  that is not realizable as the Jacobian of a bilipschitz map. That is, there is no bilipschitz  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\text{Jac}(\varphi) = \rho$  holds almost everywhere (a.e.), where  $\text{Jac}(\varphi)$  is the determinant of the Jacobian matrix of  $\varphi$ . (McMullen also showed that the existence of such a  $\rho$  is actually *equivalent* to the existence of two non-equivalent separated nets.)

According to Burago and Kleiner, the problem of density not realizable as the Jacobian of a bilipschitz map was first proposed by Moser and Reimann in the 60's. Later Dacorogna and Moser [DM90] showed that for  $\alpha \in (0, 1)$ , every  $\alpha$ -Hölder function is locally the Jacobian determinant of a  $C^{1,\alpha}$  homeomorphism, and they posed the question of whether every continuous function is locally the Jacobian of a  $C^1$  diffeomorphism. Several other authors studied the problem of prescribed Jacobian in different settings, for example Ye [Ye94] in Sobolev spaces.

McMullen's construction is simpler and a bit easier to describe than the Burago–Kleiner one. But while Burago and Kleiner provide a complete proof of the nonrealizability of their construction, McMullen's construction and its proof are only sketched in four short paragraphs, with a remark that a detailed proof can be given along the lines of the Burago–Kleiner proof.

The author of this note, as a part of his bachelor thesis [Kal12], tried to adapt the Burago–Kleiner proof to McMullen's construction, but found this less than straightforward, and ended up with modifying the Burago–Kleiner technique, introducing additional tricks, and adjusting numerical parameters of the construction. Thus, for the sake of future research interest in the details of McMullen's construction, it seems worth publishing a complete proof.

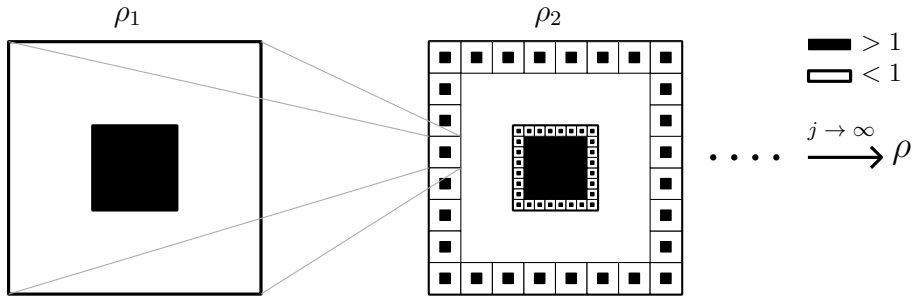


Figure 1: The first two steps of McMullen's construction.

**McMullen's construction.** The nonrealizable function  $\rho$  is actually constructed on the unit square  $S := [0, 1]^2$ , as the limit of a sequence  $\rho_1, \rho_2, \dots$  of functions, where  $\rho_j$  is obtained from  $\rho_{j-1}$  by a suitable modification.

To define  $\rho_1$ , we choose a square  $T$  of side  $\delta > 0$  at the center of  $S$  ( $\delta$  is

one of the parameters of the construction), as in Figure 1 left. We define  $\rho_1$  as a constant  $t > 1$  on  $T$  and as another constant  $s < 1$  on  $S \setminus T$ . Here  $s, t$  are chosen so that, first, the average of  $\rho$  over  $S$  is 1, and second, the image of  $T$  under any bilipschitz map with Jacobian  $\rho$  has area at least  $1 - \gamma$ , where  $\gamma > 0$  is another parameter of the construction. Here  $\gamma$  is chosen small, and thus the image of  $T$  occupies most of the image of  $S$ . McMullen chose  $\delta = 1/3$  and  $\gamma = 0.01$  in his sketch, but we will need different values.

To construct  $\rho_2$  from  $\rho_1$ , we cover each edge of the squares  $S$  and  $T$  from inside with much smaller squares; those along the edges of  $S$  have sidelength  $h_2$ , while those along the edges of  $T$  have sidelength  $\delta h_2$ , with  $h_2 > 0$  sufficiently small. We denote the collection of these new squares by  $\mathcal{S}_2$ .

On every square  $S_i \in \mathcal{S}_2$ , we define  $\rho_2$  in the same way as  $\rho_1$  was defined on  $S$ . That is, we consider a  $\delta$ -times smaller square  $T_i$  concentric with  $S_i$ , and we set  $\rho_2 = t$  on  $T_i$  and  $\rho_2 = s$  on  $S_i \setminus T_i$ ; see Figure 1 right.

On the rest of  $S$ , similar to  $\rho_1$ , the function  $\rho_2$  attains one value on the part of  $T$  not covered with  $\mathcal{S}_2$ , and another value on the part of  $S \setminus T$  not covered with  $\mathcal{S}_2$ . However, these values are slightly different from those for  $\rho_1$ , so that the average value of  $\rho_2$  on  $S$  is exactly 1, and that the area of the image of  $T$  under a bilipschitz map with Jacobian  $\rho_2$  equals exactly  $1 - \gamma$ .

The construction of  $\rho_j$  from  $\rho_{j-1}$  follows the same pattern. We cover the edges of each  $S_i \in \mathcal{S}_{j-1}$  and of the corresponding  $T_i$  from inside with squares of sidelength  $h_j$  and  $\delta h_j$ , respectively, forming a collection  $\mathcal{S}_j$ . We define  $\rho_j$  on each square of  $\mathcal{S}_j$  in the same way as  $\rho_1$  was defined on  $S$ , and we also modify the values of  $\rho_{j-1}$  on each  $S_i \in \mathcal{S}_{j-1}$  in the same way as was described above for  $\rho_2$  on  $S$ .

The sequence  $h_j$  decrease to 0 sufficiently fast, namely, so that  $h_{j-1}/h_j \rightarrow \infty$ . This ensures that the limit  $\rho = \lim_{j \rightarrow \infty} \rho_j$  is well defined a.e. in  $S$ , bounded, and also bounded away from 0. This finishes the description of McMullen's construction.

**Theorem 1 (McMullen).** *There exists no bilipschitz map  $\varphi: S \rightarrow A \subset \mathbb{R}^2$  with  $\text{Jac}(\varphi) = \rho$  a.e.*

**On differences between the Burago–Kleiner and the McMullen constructions.** The Burago–Kleiner construction provides a continuous nonrealizable function, while McMullen's construction sketched above apparently yields only a measurable one. For explaining the difference, we first describe some of the features of the Burago–Kleiner construction.

They again work in the unit square  $S$ . First, for every  $L > 1$  and  $c > 0$ , they construct a measurable function  $\rho_{L,c}: S \rightarrow [1, 1 + c]$  such that there is no  $L$ -bilipschitz homeomorphism  $\varphi: S \rightarrow \mathbb{R}^2$  with  $\text{Jac}(\varphi) = \rho_{L,c}$  a.e. The

precise construction of  $\rho_{L,c}$ , which can be found in [BK98], is not important for us at the moment.

Then they observe that if  $\{\rho_{L,c}^k\}_k$  is a sequence of smoothings of  $\rho_{L,c}$  converging to  $\rho_{L,c}$  in  $L^1$ , there must be some  $k_0 \in \mathbb{N}$  such that for every  $k \geq k_0$ , the functions  $\rho_{L,c}^k$  are also nonrealizable as Jacobians of  $L$ -bilipschitz homeomorphisms, for otherwise, the Arzelà–Ascoli theorem would yield an  $L$ -bilipschitz homeomorphism  $\varphi$  with  $\text{Jac}(\varphi) = \rho_{L,c}$  a.e.

Finally, they take a collection of disjoint squares  $S_k \subset S$  converging to a point  $p \in S$ , they construct a new function  $\rho: S \rightarrow [1, 1+c]$  by embedding the function  $\rho_{k, \min\{c, \frac{1}{k}\}}$  into  $S_k$  for every  $k \in \mathbb{N}$ , and they define  $\rho$  on the rest of  $S$  arbitrarily, while preserving its continuity. Consequently,  $\rho$  is continuous and it cannot be realized as the Jacobian of any bilipschitz homeomorphism.

Since we do not know how to prove nonrealizability of McMullen’s density parameterized so that the image of  $\rho$  is contained in  $(0, 1+c]$  with  $c > 0$  arbitrarily small, we cannot use the method of Burago and Kleiner outlined above to obtain a continuous version of McMullen’s density. However, it may be possible either to achieve continuity in some other way without changing the construction too much or devise a better proof.

## 2 Preliminaries

Before we proceed to the proof, we present some definitions and facts. We denote the  $k$ -dimensional Lebesgue measure by  $\lambda_k$ . Since we will deal mainly with the plane, we write just  $\lambda$  instead of  $\lambda_2$ . We always use the Lebesgue measure unless stated otherwise.

Let  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a map that is (Fréchet) differentiable at a point  $x \in \mathbb{R}^n$ . The matrix consisting of its first partial derivatives at  $x$  is called the *Jacobian matrix* of the map  $\varphi$  at the point  $x$ . We denote it by  $D\varphi(x)$ . The determinant of the Jacobian matrix,  $\text{Jac}(\varphi)(x) := \det D\varphi(x)$ , is called the *Jacobian determinant* or simply *Jacobian* of the map  $\varphi$  at  $x$ . It gives us information about the change of the volume in the neighborhood of  $x$ .

By a curve we mean an image of an interval  $I$  under a continuous map  $f$ . The length of a curve  $P$  is defined in the usual manner,  $\text{length}(P) := \sup \sum_{k=0}^{n-1} \|f(p_{k+1}) - f(p_k)\|$ , where the supremum is taken over all finite partitions of the form  $\min I = p_0 < p_1 < \dots < p_n = \max I$ .

Let us denote the line segment between points  $\mathbf{a}$  and  $\mathbf{b}$  by  $\overline{\mathbf{ab}}$ , the Euclidean distance between  $\mathbf{a}$  and  $\mathbf{b}$  by  $\|\mathbf{a} - \mathbf{b}\|_2$ , the projection on the first coordinate ( $x$ -axis) by  $\pi_x$ , and the restriction of a function  $f$  to a set  $E$  by  $f|_E$ .

**Observation 2.** *Let  $P$  be the image of an interval of length  $d$  under an  $L$ -Lipschitz map. Then  $\text{length}(P) \leq Ld$ .*

We leave the easy proof to the reader.

**Theorem 3.** *Let  $f: A \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k$  be an injective Lipschitz map. Then for every measurable set  $E \subseteq A$ ,  $f(E)$  is also measurable and*

$$\int_E |\text{Jac}(f)| d\lambda_k = \lambda_k(f(E)).$$

*Proof.* This theorem is a corollary of the change of variables theorem for Lebesgue integral and Rademacher's theorem; it can be found in Fremlin's monograph [Fre00, Corollary 263F], for example.  $\square$

### 3 The proof

For each square  $S_i \in \mathcal{S}_j$  we denote the smaller square placed at the center of  $S_i$  by  $T_i$ . The function  $\rho$  has the following properties. For every level  $j$  and every  $S_i \in \mathcal{S}_j$  we have  $\int_{S_i} \rho d\lambda = \lambda(S_i)$  and  $\int_{T_i} \rho d\lambda = (1 - \gamma)\lambda(S_i)$ .

*Proof of Theorem 1.* Assume to the contrary that there exists a bilipschitz homeomorphism  $\varphi: S \rightarrow \mathbb{R}^2$  with the Jacobian determinant  $\text{Jac } \varphi = \rho$  a.e.

Let us denote by  $H$  the set consisting of all edges of all covering squares, i.e., the squares in  $\bigcup_{j=1}^{\infty} \mathcal{S}_j$ , in the construction of the function  $\rho$ . We set  $K := \sup_{\mathbf{p}, \mathbf{q} \in H} \frac{\|\varphi(\mathbf{p}) - \varphi(\mathbf{q})\|_2}{\|\mathbf{p} - \mathbf{q}\|_2}$ . Since the map  $\varphi$  is bilipschitz, we get  $K < \infty$ . Now we choose a parameter  $\alpha := \alpha(\gamma, \delta, K) > 0$  sufficiently small, whose value will be set at the end of the proof. From the definition of the supremum we have that there exists a covering square  $S_i$  and one of its edges  $\overline{\mathbf{a}\mathbf{b}}$  with  $\frac{\|\varphi(\mathbf{a}) - \varphi(\mathbf{b})\|_2}{\|\mathbf{a} - \mathbf{b}\|_2} > K(1 - \alpha)$ . We fix this edge  $\overline{\mathbf{a}\mathbf{b}}$  until the end of the proof. Without loss of generality, we assume that  $\mathbf{a} = (0, 0)$ ,  $\mathbf{b} = (b, 0)$ ,  $\varphi(\mathbf{a}) = (0, 0)$ , and  $\varphi(\mathbf{b}) = (b', 0)$ , with  $b > 0$  and  $b' > K(1 - \alpha)b$ .

By the construction of  $\rho$ , the edge  $\overline{\mathbf{a}\mathbf{b}}$  is covered with arbitrarily small squares. In other words, for a chosen  $N_0 := N_0(\alpha, \gamma, \delta, K)$ , which will be set at the end of the proof, too, we can find  $N \geq N_0$  so that the edge  $\overline{\mathbf{a}\mathbf{b}}$  is covered with squares  $S_{i_1}, \dots, S_{i_N}$ .

These squares form a tiny long rectangle, which we call  $R$ . Let  $h$  stand for their sidelength; this means that  $h = b/N$ . For the sake of simplicity, we are going to denote the squares  $S_{i_1}, \dots, S_{i_N}$  just by  $S_1, \dots, S_N$ . Hence  $R = \bigcup_{i=1}^N S_i$ . Inside of every  $S_i, i = 1, \dots, N$ , we also have the square  $T_i$  with  $\delta$  times smaller sidelength. For clarity, we add Figure 2.

The main idea of the proof is that by choosing  $\alpha$  to be very small we enforce  $\varphi$  to map the long edges of the rectangle  $R$  to almost straight lines stretched by a factor almost  $K$ . The function  $\rho$  has been constructed so that most of the mass within  $S_i$  is concentrated on  $T_i$ . This implies that the majority of each  $\varphi(S_i)$  has to be filled up with  $\varphi(T_i)$ .

On the other hand, the sidelength of each  $T_i$  is  $\delta$  times smaller than that of  $S_i$ . Each side of  $T_i$  is also covered with smaller squares, and thus

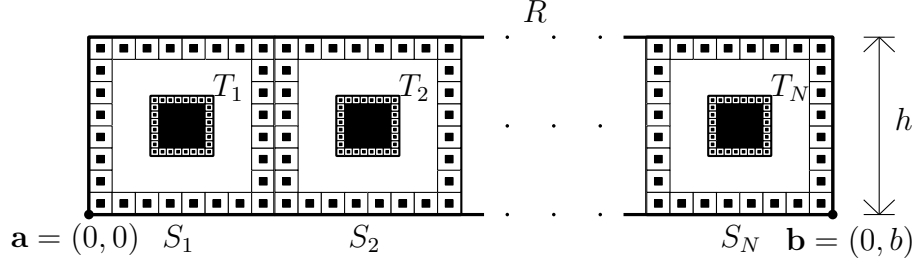


Figure 2: The rectangle  $R$  covering the edge  $\overline{\mathbf{a}\mathbf{b}}$ .

it cannot be stretched more than by the factor  $K$ . The only way in which all these conditions can be fulfilled is that the images of the long edges of the rectangle  $R$  somewhat ripple up above the images of the respective rectangles  $T_i$  while between them they have to ripple down. But this forces the images of the long edges of  $R$  to become very long, eventually longer than the constant  $K$  allows, which leads to a contradiction. We illustrate the outlined idea in Figure 3.

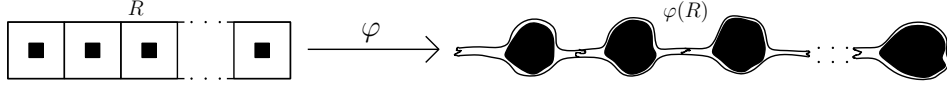


Figure 3: Deformation of the edges of  $R$  under the map  $\varphi$ .

In order to make this idea rigorous, we look at the change of the length of vertical cuts through the set  $\varphi(R)$ . The most complicated part of the proof is an estimation of the length of the boundary of  $\varphi(R)$ . Because we have almost no control of  $\varphi$  locally, the shapes of different  $\varphi(S_i)$  can be various. We manage the described difficulty by examining the squares  $S_i$  in seven-tuples. For this purpose we make an additional technical assumption that the number of squares covering every edge during the construction of  $\rho$  is divisible by seven. Let  $R_i$  stand for the rectangle formed by the seven consecutive squares  $S_{7i-6}, \dots, S_{7i}$ . Without loss of generality, we assume that the edges of the squares  $R_i$  covering the edge  $\overline{\mathbf{a}\mathbf{b}}$  are their bottom edges. Let us write  $\mathbf{a}_i$  and  $\mathbf{b}_i$  for the left and right bottom vertices of the rectangle  $R_i$ , respectively, and  $\mathbf{c}_i$  and  $\mathbf{d}_i$  for the left and right upper vertices of  $R_i$ , respectively.

**Definition 4.** We call the rectangle  $R_i$  with its bottom vertices  $\mathbf{a}_i$  and  $\mathbf{b}_i$  *nice* if  $|\pi_x(\varphi(\mathbf{a}_i)) - \pi_x(\varphi(\mathbf{b}_i))| > K(1 - 2\alpha)7h$ .

The factor  $1 - 2\alpha$  in the preceding definition is chosen to have a constant fraction of the rectangles  $R_i$  nice, more precisely, to get the following observation:

**Observation 5.** There are at least  $N/14$  nice rectangles  $R_i$ .

*Proof.* Let  $r$  stand for the number of *nice* rectangles  $R_i$ . This implies that for  $N/7 - r$  rectangles  $R_i$  we have  $|\pi_x(\varphi(\mathbf{a}_i)) - \pi_x(\varphi(\mathbf{b}_i))| \leq K(1 - 2\alpha)7h$ . On the other hand, the edges  $\overline{\mathbf{a}_i\mathbf{b}_i}$  connect the vertices  $\mathbf{a}$  and  $\mathbf{b}$ . The horizontal distance of the points  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$  must be greater than  $K(1 - \alpha)Nh$ . Since no edge  $\overline{\mathbf{a}_i\mathbf{b}_i}$  can be stretched more than by the factor  $K$ , we can calculate a lower bound on  $r$ :

$$\left(\frac{N}{7} - r\right)(K(1 - 2\alpha)7h) + Kr \cdot 7h > K(1 - \alpha)Nh$$

Simple calculation yields  $r > N/14$ .  $\square$

Now we define the set  $D_i := \pi_x(\varphi(\overline{\mathbf{a}_i\mathbf{b}_i})) \cap \pi_x(\varphi(\overline{\mathbf{c}_i\mathbf{d}_i}))$  for every rectangle  $R_i$ . Next, we define a function  $f_i: D_i \rightarrow [0, \infty)$  measuring the length of vertical cuts through the set  $\varphi(R_i)$  at a point  $x \in D_i$ , i.e.,  $f_i(x) = \lambda_1(\{y \in \mathbb{R} \mid (x, y) \in \varphi(R_i)\})$ . It follows from the continuity of the map  $\varphi^{-1}$  that the function  $f_i$  is Lebesgue integrable for every  $R_i$ .

We would like to stress that the function  $f_i$  is measuring only the length of cuts through the set  $R_i$ , not the entire  $R$ . Vertical cuts through  $R$  can intersect many  $R_i$ 's.

The following observation will later help us treat each *nice* rectangle separately. It basically says, that the image of every such an  $R_i$  is drawn almost horizontally and its boundary is not “too wavy”.

**Observation 6.** *We have  $\pi_x(\varphi(S_{7i-3})) \cap \pi_x(\varphi(\overline{\mathbf{a}_i\mathbf{c}_i})) = \emptyset$ , and symmetrically,  $\pi_x(\varphi(S_{7i-3})) \cap \pi_x(\varphi(\overline{\mathbf{b}_i\mathbf{d}_i})) = \emptyset$  for every nice rectangle  $R_i$ .*

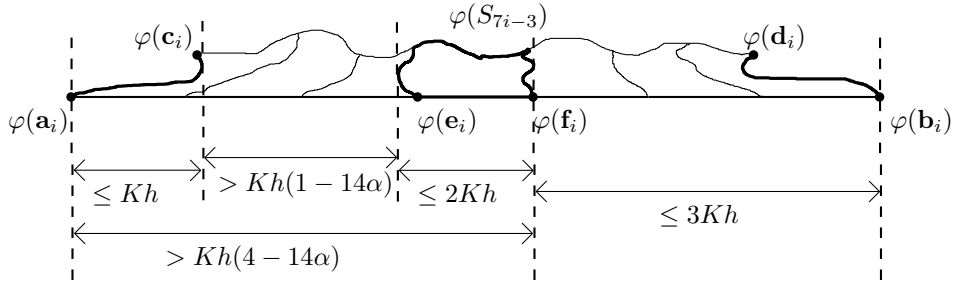


Figure 4: The image of  $R_i$  under  $\varphi$ .

*Proof.* Let  $\mathbf{e}_i$  stand for the left bottom vertex of  $S_{7i-3}$  and  $\mathbf{f}_i$  for its right bottom vertex. This implies  $\|\mathbf{a}_i - \mathbf{f}_i\|_2 = 4h$  and  $\|\mathbf{b}_i - \mathbf{f}_i\|_2 = 3h$ . Since the rectangle  $R_i$  is *nice*, we also have  $|\pi_x(\varphi(\mathbf{a}_i)) - \pi_x(\varphi(\mathbf{f}_i))| > K(1 - 2\alpha)7h - 3Kh = Kh(4 - 14\alpha)$ .

The set  $\varphi(S_{7i-3})$  has to lie within the circle with radius  $2Kh$  centered at  $\varphi(\mathbf{f}_i)$ , while the set  $\varphi(\overline{\mathbf{a}_i\mathbf{c}_i})$  lies within the circle of radius  $Kh$  centered at  $\varphi(\mathbf{a}_i)$ . We conclude that the distance between the sets  $\pi_x(\varphi(S_{7i-3}))$  and

$\pi_x(\varphi(\overline{\mathbf{a}_i \mathbf{c}_i}))$  is at least  $Kh(4 - 14\alpha) - 2Kh - Kh = Kh(1 - 14\alpha)$ , which is positive for  $\alpha \in (0, 1/14)$ . This proof is outlined in Figure 4.

The second part of the observation is obtained symmetrically.  $\square$

The rectangle  $R_i$  is formed by the squares  $S_{7i-6}, \dots, S_{7i}$ . We define the sets  $V_i := \pi_x(\varphi(T_{7i-3}))$  and  $C_i := \pi_x(\varphi(S_{7i-3})) \setminus \pi_x(\bigcup_{j=7i-6}^{7i} \varphi(T_j))$ ; so we have  $V_i \cap C_i = \emptyset$ . By Observation 6 it is clear that  $\pi_x(\varphi(S_{7i-3})) \subseteq D_i$ , and thus  $C_i, V_i \subseteq D_i$  for *nice*  $R_i$ .

Observation 6 implies that, whenever  $R_i$  is *nice*, for every  $x \in V_i \cup C_i$ , all the points of intersection of the vertical cut at  $x$  with the boundary of  $\varphi(R_i)$  also lie on the boundary of  $\varphi(R)$ . Indeed, since the map  $\varphi$  is a homeomorphism, the images of the long edges of  $R_i$ , which form a part of the boundary of  $\varphi(R_i)$ , are also part of the boundary of  $\varphi(R)$ . We will use these points to bound the length of the boundary of  $\varphi(R)$ .

It is possible that the sets  $V_i, V_{i+1}, C_i$ , and  $C_{i+1}$  are not mutually disjoint. Let  $x$  be a common point of  $C_i$  and  $C_{i+1}$ , for example. We already know that, if  $R_i$  and  $R_{i+1}$  are both *nice*, the points of intersection of  $\varphi(R_i)$  and  $\varphi(R_{i+1})$  with the vertical cut at  $x$  lie on the boundary of  $\varphi(R)$ . The fact that  $\varphi$  is a homeomorphism implies that all these points are different. This is a crucial observation in our proof, because it allows us to do the estimates for each of the *nice* rectangles separately and then sum them up.

Let us set  $v_i := \lambda_1(V_i)$  and  $c_i := \lambda_1(C_i)$ . Since the length of the edge of  $T_i$  is  $\delta h$ , we have that the length of the boundary of  $\varphi(T_{7i-3})$  is at most  $4K\delta h$ , which implies  $\lambda_1(\pi_x(\varphi(T_{7i-3}))) \leq 2K\delta h$ , and hence  $v_i \leq 2K\delta h$  for every  $i \in [N/7]$ .

For every *nice*  $R_i$  we have  $\lambda_1(\pi_x(\varphi(S_{7i-3}))) > K(1 - 2\alpha)7h - 6Kh = Kh(1 - 14\alpha)$ , and thus  $c_i > Kh(1 - 14\alpha) - 7 \cdot 2K\delta h = Kh(1 - 14\alpha - 14\delta)$ . Because we need  $c_i > 0$ , we have to choose  $\delta \in (0, 1/14)$  and  $\alpha \in (0, (1 - 14\delta)/14)$ .

By Theorem 3 and since we assume  $\text{Jac}(\varphi) = \rho$  a.e., we have  $\lambda(\varphi(T_{7i-3})) = (1 - \gamma)h^2$ . We define two constants  $h_V^i$  and  $h_C^i$  denoting the average value of  $f_i$  over the sets  $V_i$  and  $C_i$ , respectively. In other words, the following holds:

$$\lambda(\varphi(T_{7i-3})) < \int_{V_i} f_i d\lambda_1 =: h_V^i \cdot v_i$$

$$\lambda\left(\bigcup_{j=7i-6}^{7i} \varphi(S_j) \setminus \bigcup_{j=7i-6}^{7i} \varphi(T_j)\right) > \int_{C_i} f_i d\lambda_1 =: h_C^i \cdot c_i.$$

The upper bound on  $v_i \leq 2K\delta h$  yields

$$h_V^i > \frac{\lambda(\varphi(T_{7i-3}))}{v_i} \geq \frac{(1 - \gamma)h^2}{2K\delta h} = \frac{1}{K} \cdot \frac{1 - \gamma}{2\delta} h.$$



For every *nice*  $R_i$  and by the lower bound on  $c_i > Kh(1 - 14\alpha - 14\delta)$  the following holds:

$$\begin{aligned} h_C^i &< \frac{\lambda(\bigcup_{j=7i-6}^{7i} \varphi(S_j) \setminus \bigcup_{j=7i-6}^{7i} \varphi(T_j))}{c_i} < \frac{7\gamma \cdot h^2}{Kh(1 - 14\alpha - 14\delta)} \\ &= \frac{1}{K} \cdot h \cdot \left( \frac{7\gamma}{1 - 14\alpha - 14\delta} \right). \end{aligned}$$

Let us set  $\Delta := h_V^i - h_C^i$ . Since  $h_V^i$  and  $h_C^i$  are the average values of  $f_i$  over  $V_i$  and  $C_i$ , respectively, we get that for every *nice*  $R_i$  there must be two points  $x \in V_i$  and  $y \in C_i$  such that  $f_i(x) \geq h_V^i$  and  $f_i(y) \leq h_C^i$ . Thus we have  $f_i(x) - f_i(y) \geq h_V^i - h_C^i = \Delta$ .

Now we would like to argue that  $\Delta$  is the lower bound on the change of height of  $\varphi(R)$  over  $V_i \cup C_i$  for every *nice*  $R_i$ . Indeed, it is true that there are two points  $\mathbf{u}_1, \mathbf{u}_2$ , one from the image of the bottom edge  $\overline{\mathbf{a}_i \mathbf{b}_i}$  of  $R_i$ , the other from the image of the upper edge  $\overline{\mathbf{c}_i \mathbf{d}_i}$ , such that  $\mathbf{u}_1 = (x, u_1)$ ,  $\mathbf{u}_2 = (x, u_2)$  and  $u_2 - u_1 \geq h_V^i$ . But the same thing about  $y$  and  $h_C^i$  has to be said with a little more care.

The problem is that the vertical cut through  $\varphi(R_i)$  at  $y$  does not have to be connected, i.e., it may consist of several line segments even for *nice*  $R_i$ . But we know that the length of these line segments is at most  $h_C^i$  in total, and hence every line segment of this cut is at most  $h_C^i$  long. Consequently, we infer that there are two points  $\mathbf{l}_1 \in \varphi(\overline{\mathbf{a}_i \mathbf{b}_i})$ ,  $\mathbf{l}_2 \in \varphi(\overline{\mathbf{c}_i \mathbf{d}_i})$  such that  $\mathbf{l}_1 = (y, l_1)$ ,  $\mathbf{l}_2 = (y, l_2)$  and  $|l_2 - l_1| \leq h_C^i$ . See Figure 5.

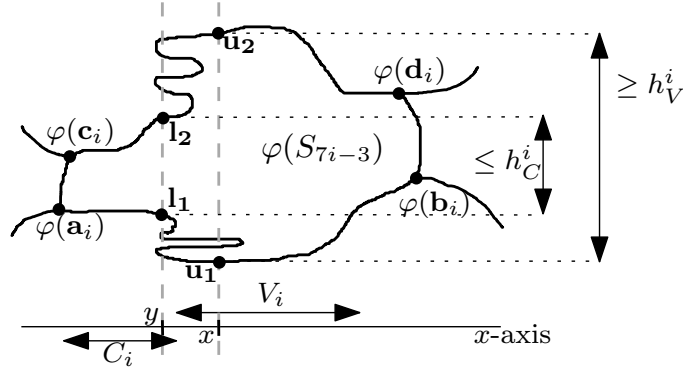


Figure 5: The lower bound on the change of height of  $\varphi(R_i)$ .

By Observation 5 and the discussion above, it follows that the overall change in height of  $\varphi(R)$  is at least  $\frac{\Delta N}{14}$ .

**Observation 7.** For every point  $x \in \varphi(\overline{\mathbf{a} \mathbf{b}})$  the distance between the point  $x$  and the line segment  $\overline{\varphi(\mathbf{a}) \varphi(\mathbf{b})}$  is less than  $Kb/2 \cdot \sqrt{\alpha(2 - \alpha)}$ .

*Proof.* By Observation 2 we know that the length of the curve  $\varphi(\overline{\mathbf{a} \mathbf{b}})$  is at most  $Kb$ . Since  $\varphi(\overline{\mathbf{a} \mathbf{b}})$  also connects the points  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$ ,  $\varphi(\overline{\mathbf{a} \mathbf{b}})$

has to lie inside the ellipse with two focal points  $\varphi(\mathbf{a}), \varphi(\mathbf{b})$  and sum of the distances from any point on the ellipse to its foci equal to  $Kb$ .

We can calculate the upper bound on the length of the semi-minor axis of this ellipse, because we know that  $\|\varphi(\mathbf{a}) - \varphi(\mathbf{b})\|_2 > Kb(1 - \alpha)$ ; using the Pythagorean theorem we have that the length of the semi-minor axis is less than  $\sqrt{(Kb/2)^2 - (Kb(1 - \alpha)/2)^2} = Kb/2 \cdot \sqrt{\alpha(2 - \alpha)}$ .  $\square$

Let us denote by  $P$  the other part of the boundary of  $\varphi(R)$  not containing  $\varphi(\overline{\mathbf{ab}})$ . We define  $\Omega := \frac{\Delta N}{14} - Kb/2 \cdot \sqrt{\alpha(2 - \alpha)}$  denoting the lower bound on the sum of vertical distances over all *nice* rectangles, which has to be overcome by  $P$  between the points  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$ . We have supposed that  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$  lie on the  $x$ -axis, and moreover, we know that  $\|\varphi(\mathbf{a}) - \varphi(\mathbf{b})\|_2 > Kb(1 - \alpha)$ .

In order to compute the lower bound on  $\text{length}(P)$ , we can argue as follows. Since  $\Omega$  is the lower bound on the sum of changes of vertical distances made by  $P$ , it has to be possible to draw a curve of length equal to  $\text{length}(P)$  that goes from  $\varphi(\mathbf{a})$  to a point lying at least  $\Omega/2$  higher than  $\varphi(\mathbf{a})$  and then go down to  $\varphi(\mathbf{b})$ . Such a curve is the shortest when it forms the legs of an isosceles triangle of height  $\Omega/2$ . Again, using the Pythagorean theorem, as illustrated in Figure 6, we conclude

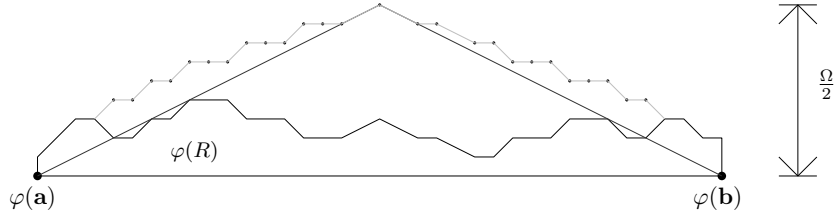


Figure 6: The lower bound on  $\text{length}(P)$ .

$$\text{length}(P) > 2\sqrt{\left(\frac{\Omega}{2}\right)^2 + \left(\frac{Kb(1 - \alpha)}{2}\right)^2} = \sqrt{\Omega^2 + (Kb(1 - \alpha))^2}.$$

The curve  $P$  consists of the images of  $N + 2$  line segments of length  $h$ , and thus  $\text{length}(P) \leq K(N + 2)h$ :

$$\Omega^2 + K^2b^2(1 - \alpha)^2 \leq K^2(N + 2)^2h^2$$

Substituting for  $\Omega = \frac{\Delta N}{14} - Kb/2 \cdot \sqrt{\alpha(2 - \alpha)}$  and for  $b = Nh$ , we obtain, with some calculations,

$$\frac{\Delta^2 N^2}{196} - \frac{\Delta N^2}{14} Kh \sqrt{\alpha(2 - \alpha)} \leq K^2 h^2 (4N + 4 + \frac{3N^2}{4} \alpha(2 - \alpha)).$$

Let us define  $q = q(\alpha, \gamma, \delta) := \frac{1-\gamma}{2\delta} - \frac{7\gamma}{1-14\alpha-14\delta}$ . We substitute for  $\Delta > \frac{1}{K}hq$ :

$$\begin{aligned} \frac{h^2 q^2 N^2}{196K^2} - \frac{Kh^2 q N^2}{14K} \sqrt{\alpha(2-\alpha)} &\leq K^2 h^2 (4N + 4 + \frac{3N^2}{4} \alpha(2-\alpha)) \\ q^2 &\leq 196K^4 \left( \frac{4}{N} + \frac{4}{N^2} + \frac{3}{4} \alpha(2-\alpha) \right) + 14qK^2 \sqrt{\alpha(2-\alpha)}. \end{aligned}$$

From the last inequality we can see, provided  $q$  is positive, that for  $N \rightarrow \infty$  and  $\alpha \rightarrow 0$  the right hand side of the inequality converges to zero, while  $q$  slightly grows up to its limit value  $\frac{1-\gamma}{2\delta} - \frac{7\gamma}{1-14\delta}$ , eventually leading to a contradiction. The positivity of  $q$  can clearly be ensured by a suitable setting of the constants  $\delta, \gamma$ , and  $\alpha$ .

□

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